

THEORY OF PARTITION CHROMATOGRAPHY. IV

HANS VINK

Institute of Physical Chemistry, University of Uppsala (Sweden)

(Received March 28th, 1966)

In this article the theory of partition chromatography from earlier work¹⁻³ is generalized by using a stepwise approach in the diffusion model treatment of the partition process. In the earlier theory, the diffusion model⁴ was applied as a first order approximation, as it considered a single stationary phase of uniform concentration, and thus did not take into account the solute distribution in that phase. If, for example, the column filling consists of spherical beads, a rigorous treatment must take into account the time-dependent radial concentration distribution in a sphere, resulting from diffusion between the sphere and the surrounding medium. Although diffusion into a sphere may be treated rigorously for some special cases⁵, the resulting formulae have the form of slowly convergent infinite series which are unsuitable for use in chromatographic theory. However, by generalizing the diffusion model the rigorous treatment may be approached and the theory of chromatography brought to a high degree of perfection. In the generalized model, the beads are divided into concentric shells and the diffusion model applied to consecutive shells, the concentration in each shell being taken as uniform. Thus, in analogy with the earlier model, each shell is considered as constituting a separate phase, and assigned its own concentration variable. An essential feature of the model is the fact that direct exchange of solute can only take place between two adjacent phases. A pictorial representation of the model is shown in Fig. 1.

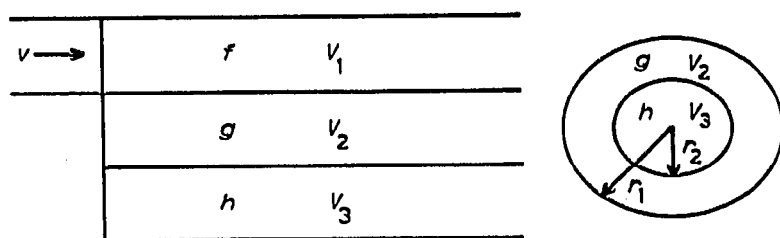


Fig. 1. Schematic representation of a three-step column model.

For this model the differential equations for column operation may now be written down. In order to simplify the treatment, at first only three phases are considered, the generalization to an arbitrary number of phases being undertaken later. Also, longitudinal diffusion in the stationary phase is omitted, as in most practical cases it is incompatible with the column model (this is the case when the column filling consists of particles with small dimensions in the axial direction of the column, as the particles are then surrounded by a solution of uniform concentration). In

other respects the model is completely general and takes into account variations in partition and diffusion coefficients within the particles. With these premises the following equations are obtained:

$$\frac{\partial f}{\partial t} = D_1 \frac{\partial^2 f}{\partial x^2} - v \frac{\partial f}{\partial x} + \frac{a_{12}}{V_1} (g - \gamma_{12}f) \quad (1)$$

$$\frac{\partial g}{\partial t} = -\frac{a_{12}}{V_2} (g - \gamma_{12}f) + \frac{a_{23}}{V_2} (h - \gamma_{23}g) \quad (2)$$

$$\frac{\partial h}{\partial t} = -\frac{a_{23}}{V_3} (h - \gamma_{23}g) \quad (3)$$

The symbols have the following meaning:

- f = concentration of solute in the mobile phase;
- g = concentration of solute in the first shell of the stationary phase;
- h = concentration of solute in the second shell of the stationary phase;
- v = translational velocity of the mobile phase;
- D_1 = diffusion coefficient in the mobile phase;
- V_1, V_2, V_3 = volumes of the different phases per unit of interphase area between the mobile and stationary phase;
- γ_{12}, γ_{23} = partition coefficients between the different phases;
- a_{12}, a_{23} = diffusion factors between different phases.

The factors a_{ij} warrant some discussion here. The factor a_{12} , referring to diffusion between the mobile phase and the first stationary phase, is obviously unchanged from the earlier theory. Hence:

$$a_{12} = \frac{2 D_2}{V_2} \quad (4)$$

For a_{23} , however, some new features appear. In the first place, rate determining diffusion now occurs in both phases, with diffusion coefficients D_2 and D_3 respectively. In the second place the capacity elements in the diffusion model are the volumes of the respective phases per unit of interphase area. They are here designated V_{23} and V_{32} and are obviously different from V_2 and V_3 , which are the per area volumes with respect to the first phase boundary. A value for a_{23} may be constructed by analogy to eqn. (4) by taking averages for both V_{23}, V_{32} and D_2, D_3 . Furthermore, the factor 2 is omitted because the rate determining diffusion occurs in both phases. Hence:

$$a_{23} = \frac{D_2 + D_3}{V_{23} + V_{32}} \quad (5)$$

Although it is derived in a somewhat arbitrary manner, it is felt that this formula gives a reasonably good approximation of diffusion between adjacent shells in a particle. Obviously the approximation is better the smaller the spacing between the shells.

APPLICATION OF THE MOMENT METHOD

Equations (1)–(3) may be solved by the application of the moment method³. The i -th moment of the concentration distribution in the different phases is defined as follows:

$$F_i = \int_a^b x^i f dx; \quad G_i = \int_a^b x^i g dx; \quad H_i = \int_a^b x^i h dx \quad (6)$$

where a and b are points on either side of the peak in regions where the concentration is zero.

Following the procedure in ref. 3 we get for the first three moments the following systems of differential equations.

Zeroth moment:

$$\dot{F}_0 - \frac{a_{12}}{V_1} (G_0 - \gamma_{12} F_0) = 0 \quad (7)$$

$$\dot{G}_0 + \frac{a_{12}}{V_2} (G_0 - \gamma_{12} F_0) - \frac{a_{23}}{V_2} (H_0 - \gamma_{23} G_0) = 0 \quad (8)$$

$$\dot{H}_0 + \frac{a_{23}}{V_3} (H_0 - \gamma_{23} G_0) = 0 \quad (9)$$

First moment:

$$\dot{F}_1 - \frac{a_{12}}{V_1} (G_1 - \gamma_{12} F_1) = v F_0 \quad (10)$$

$$\dot{G}_1 + \frac{a_{12}}{V_2} (G_1 - \gamma_{12} F_1) - \frac{a_{23}}{V_2} (H_1 - \gamma_{23} G_1) = 0 \quad (11)$$

$$\dot{H}_1 + \frac{a_{23}}{V_3} (H_1 - \gamma_{23} G_1) = 0 \quad (12)$$

Second moment:

$$\dot{F}_2 - \frac{a_{12}}{V_1} (G_2 - \gamma_{12} F_2) = 2 D_1 F_0 + 2 v F_1 \quad (13)$$

$$\dot{G}_2 + \frac{a_{12}}{V_2} (G_2 - \gamma_{12} F_2) - \frac{a_{23}}{V_2} (H_2 - \gamma_{23} G_2) = 0 \quad (14)$$

$$\dot{H}_2 + \frac{a_{23}}{V_3} (H_2 - \gamma_{23} G_2) = 0 \quad (15)$$

These equations are first order linear differential equations with constant coefficients and may be solved by standard methods. The general solution is obtained

by adding the solution of the corresponding homogeneous system to a particular solution of the system. We are here only concerned with the steady state solutions K_t , L_t and M_t , which are obtained by deleting the exponential terms in the general solution. They are readily determined by the method of undetermined coefficients.

The equation system for the zeroth moment is homogeneous, hence K_0 , L_0 and M_0 are constants. Substituting into (7)–(9) and using the material balance eqn. (16),

$$V_1 K_0 + V_2 L_0 + V_3 M_0 = C \quad (16)$$

we get:

$$L_0 = \gamma_{12} K_0 \quad (17)$$

$$M_0 = \gamma_{23} L_0 \quad (18)$$

$$K_0 = \frac{C}{V_1 + \gamma_{12}(V_2 + \gamma_{23}V_3)} \quad (19)$$

The first moments are now obtained from eqns. (10)–(12), where F_0 in (10) is replaced by K_0 from (19). The solutions may be written in the form:

$$K_1 = k_1 t + k \quad (20)$$

$$L_1 = l_1 t + l \quad (21)$$

$$M_1 = m_1 t + m \quad (22)$$

where the constants on the right hand side of the equations are evaluated by insertion into (10)–(12) and identification of the coefficients. The following results are obtained, expressed in terms of the reduced moments for the respective phases:

$$\mu_f = \frac{K_1}{K_0} = \frac{v}{1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1}} t + C' \quad (23)$$

$$\mu_g = \frac{L_1}{L_0} = \mu_f - \frac{v(V_2 + \gamma_{23}V_3)}{a_{12} \left(1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1}\right)} \quad (24)$$

$$\mu_h = \frac{M_1}{M_0} = \mu_g - \frac{vV_3}{a_{23} \left(1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1}\right)} \quad (25)$$

where C' is the integration constant.

Finally, the second moments are obtained from eqns. (13)–(15), where F_0 and F_1 in (13) are replaced by K_0 and K_1 from (19) and (23) respectively. Here the solutions take the form:

$$K_2 = k'_2 t^2 + k'_1 t + k' \quad (26)$$

$$L_2 = l'_2 t^2 + l'_1 t + l' \quad (27)$$

$$M_2 = m'_2 t^2 + m'_1 t + m' \quad (28)$$

and the constants on the right hand side are evaluated as above. Here the result is stated in the form of the variance for the concentration distribution in the mobile phase, which may be written in the form:

$$\mu_2 = \frac{K_2}{K_0} - \frac{K_1^2}{K_0^2} = 2Dt + \text{const.} \quad (29)$$

where:

$$D = D' + D'' = \frac{D_1}{1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1}} + \frac{v^2 \gamma_{12}}{V_1 \left(1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1} \right)^3} \cdot \left\{ \frac{(V_2 + \gamma_{23}V_3)^2}{a_{12}} + \frac{\gamma_{23}V_3^2}{a_{23}} \right\} \quad (30)$$

The formulae for a general n -step model may now be obtained from the above formulae by induction. In all formulae we have to make the successive substitutions:

$$V_k \longrightarrow V_k + \gamma_{k, k+1} V_{k+1} \text{ for } k = 3, \dots, n-1 \quad (31)$$

Further, for every step the expression in the brackets of (30) has to be supplemented with the term:

$$\frac{\gamma_{k, k+1} V_{k+1}^2}{a_{k, k+1}} \text{ for } k = 3, \dots, n-1 \quad (32)$$

CONCLUSIONS

The laws of chromatography may now be formulated for the generalized partition chromatography column.

Beginning with the zeroth moments, which determine the amount of solute in the different phases, we find them to be constants, independent of the velocity of the mobile phase. Thus, under steady state conditions the solute distribution between the different phases is the same as under static equilibrium conditions.

The first moments determined the location of the peaks in the chromatographic column. The peaks in the different phases are displaced relative to each other, the magnitudes of the displacements being given by eqns. (24) and (25). For all peaks the relative peak velocity ν has the value:

$$\nu = \frac{\dot{\mu}}{v} = \frac{1}{1 + \gamma_{12} \frac{V_2 + \gamma_{23}V_3}{V_1}} \quad (33)$$

A comparison with (19) shows that ν is simply the fraction of solute in the mobile phase $V_1 K_0 / C$. The laws governing the behaviour of the zeroth and first moments may thus be stated independently of the model and are to be considered as fundamental laws of chromatography.

The variance is a measure of the width of a peak. It increases linearly with time and may be expressed in terms of a spreading coefficient D , eqn. (29). It consists of two terms. The first, D' , is due to longitudinal diffusion in the mobile phase and equals the diffusion coefficient multiplied by the fraction of solute in the mobile phase. Even this term is independent of the column model. The second term, D'' , is the chromatographic dispersion. Here the generalized model yields a result different from the original model. We will consider this difference more closely for the special case of a column filling consisting of spherical particles. To specify the model we consider particles of uniform size and material. Then:

$$\gamma_{12} = \gamma$$

$$\gamma_{23} = 1$$

$$D_2 = D_3$$

If we divide the particles into two concentric shells and denote the radius of the particle by r_1 and that of the core by r_2 and use the symbols V'_1 and V'_2 for the volume parameters of the original model, we get

$$V_1 = V'_1$$

$$V_2 = \left(\frac{r_2}{r_1}\right)^2 V_{23}$$

$$V_3 = \left(\frac{r_2}{r_1}\right)^2 V_{32}$$

$$V_2 + V_3 = V'_2 = \frac{r_1}{3}$$

Further we have:

$$V_3 = \frac{4/3 \pi r_2^3}{4 \pi r_1^2} = \left(\frac{r_2}{r_1}\right)^3 \frac{r_1}{3} = \left(\frac{r_2}{r_1}\right)^3 V'_2$$

Inserting from these relations into the expression for D'' in eqn. (30), together with (4) and (5), we get:

$$D'' = \frac{v^2 \gamma V'_2{}^3}{2 D_2 V'_1 \left(1 + \gamma \frac{V'_2}{V'_1}\right)^3} \left\{ \left(\frac{r_2}{r_1}\right)^4 - \left(\frac{r_2}{r_1}\right)^3 + 1 \right\} \quad (34)$$

In eqn. (34), the first factor is the chromatographic dispersion for the original model and the second factor is unity for $r_2 = 0$ and $r_2 = r_1$, and has the minimum value 0.8945 for $r_2/r_1 = 0.75$.

We will next extend the treatment to the general n -step model and evaluate D'' in the limit as n tends to infinity. We now have the relations:

$$\gamma_{12} = \gamma$$

$$\gamma_{23} = \dots = \gamma_{n-1, n} = \mathbf{I}$$

$$D_2 = D_3 = \dots = D_n$$

$$V_1 = V'_1$$

$$V_k = \left(\frac{r_k}{r_1}\right)^2 V_{k, k+1} \text{ for } k = 2, \dots, n-1$$

and:

$$\sum_{k=2}^n V_k = V'_2 = \frac{r_1}{3}$$

With these relations we get the following expression for D'' :

$$D'' = \frac{v^2 \gamma}{V'_1 \left(1 + \gamma \frac{V'_2}{V_1}\right)^3} \left\{ \frac{\left(\sum_2^n V_i\right)^2}{a_{12}} + \dots + \frac{\left(\sum_{k+1}^n V_i\right)^2}{a_{k, k+1}} + \dots + \frac{V_n^2}{a_{n-1, n}} \right\} \quad (35)$$

To evaluate the series in the parenthesis we observe that:

$$\sum_{k+1}^n V_i = \frac{r_1}{3} \left(\frac{r_k}{r_1}\right)^3 = V'_2 \left(\frac{r_k}{r_1}\right)^3 \quad (36)$$

and:

$$a_{k, k+1} = \frac{2 D_2}{(V_k + V_{k+1}) \left(\frac{r_1}{r_k}\right)^2} \quad (37)$$

Further, using equal spacing for consecutive shells, with $r_{k-1} - r_k = \varepsilon$, we get:

$$V_k = \frac{4/3 \pi (r_{k-1}^3 - r_k^3)}{4 \pi r_1^2} \approx \left(\frac{r_k}{r_1}\right)^2 \cdot \varepsilon \quad (38)$$

where the last relation holds as a first order approximation in ε . Thus:

$$a_{k, k+1} \approx \frac{D_2}{\varepsilon}$$

and the general term in the series becomes:

$$\frac{\left(\sum_{k+1}^n V_i\right)^2}{a_{k, k+1}} = \frac{V'_2{}^2}{D_2 r_1^6} r_k^6 \varepsilon \quad (39)$$

Taking the sum of the series and passing to limits, we get:

$$\frac{V'_2{}^2}{D_2 r_1^6} \sum_{k=1}^{n-1} r_k^6 \varepsilon \underset{\substack{\varepsilon \rightarrow 0 \\ n \rightarrow \infty}}{=} \frac{V'_2{}^2}{D_2 r_1^6} \int_0^{r_1} r^6 dr = \frac{3 V'_2{}^3}{7 D_2} \quad (40)$$

Hence:

$$D'' = \frac{v^2 \gamma V'_2{}^3}{2 D_2 V'_1 \left(1 + \gamma \frac{V'_2}{V'_1}\right)^3} \cdot 0.7 \quad (41)$$

We thus find that the generalized model yields a somewhat smaller value for the chromatographic dispersion than the original single-step model. The deviation is small, however, being given by the factor 0.8945 in the case of a two-step model, and by the factor 0.8571 for the continuous model. This demonstrates the rapid convergence of the results of the finite-step model and indicates that the original single-step model is an excellent approximation.

Eqn. (30) may be used also to investigate the influence of film diffusion on the chromatographic process. Then the outermost shell is composed of pure solvent. This case is somewhat more complicated since V_1 is affected by the solvent shell.

FRONTAL ANALYSIS

So far we have only considered the behaviour of isolated peaks in a chromatographic column. As a remarkable extension of the theory we will now show that the above formulae are also valid in the case of frontal analysis, if the moments are defined for concentration gradients, instead of concentration distributions. Thus, in this case we define:

$$F_t = \int_a^b x^t \frac{\partial f}{\partial x} dx; G_t = \int_a^b x^t \frac{\partial g}{\partial x} dx; H_t = \int_a^b x^t \frac{\partial h}{\partial x} dx \quad (42)$$

where a and b are points in the plateau regions on either side of the gradient peak.

Taking derivatives with respect to time and assuming exchangeability in the order of derivation:

$$\frac{\partial^2 f}{\partial t \partial x} = \frac{\partial^2 f}{\partial x \partial t} \quad (43)$$

we get:

$$\dot{F}_t = \int_a^b x^t \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial t} \right) dx; \dot{G}_t = \int_a^b x^t \frac{\partial}{\partial x} \left(\frac{\partial g}{\partial t} \right) dx; \dot{H}_t = \int_a^b x^t \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial t} \right) dx \quad (44)$$

The derivatives with respect to time of f , g and h may now be introduced from eqns. (1)–(3) and after partial integration eqns. (7)–(15) are reproduced.

SUMMARY

A generalized theory of partition chromatography is presented. It takes into account the lateral solute distribution in the stationary phase by using a stepwise approach in the treatment of the partition process. From this the continuous model is obtained in the limit of infinitesimal steps. The resulting differential equations are

solved by the moment method for steady state conditions. For the zeroth and first moments the results are shown to be independent of the specified model, whereas in the case of the second moment the generalized model yields a smaller spreading coefficient than the original single-step model. The moment method is shown to be equally applicable to frontal analysis, if the moments are defined in terms of concentration gradients in this case.

REFERENCES

- 1 H. VINK, *J. Chromatog.*, 15 (1964) 488.
- 2 H. VINK, *J. Chromatog.*, 18 (1965) 25.
- 3 H. VINK, *J. Chromatog.*, 20 (1965) 305.
- 4 H. VINK, *Acta Chem. Scand.*, 18 (1964) 409.
- 5 J. CRANK, *The Mathematics of Diffusion*, Clarendon Press, Oxford, 1956.

J. Chromatog., 25 (1966) 71-79